

Higher Helicity of Magnetic Lines and Arf-invariants

Petr M. Akhmetiev

IZMIRAN, Troitsk, Moscow

Introduction

V.I.Arnold formulated the following problem [[Arn], Problem 1984- 12]: "To transform the asymptotic ergodic definition of the Hopf invariant of divergence-free vector fields to the theory of S.P.Novikov which generalizes the Whitehead product of homotopy groups of spheres". In the paper we recall and simplify (a partial) solution of the problem from the [A4] and present new results, which generalize the problem to non-simply connected manifolds.

In the first section, we present an additional motivation of the Arnold Problem, which is based on mean magnetic field theory. We use geometrical considerations due to K.Moffatt and formulate properties of invariants in ideal MHD, which are asymptotic and ergodic properties.

Then we recall the definition of the quadratic helicity invariant and of the higher asymptotic ergodic M -invariant. We present a simpler new proof (in part) that the M -invariant is ergodic. The M -invariant is a higher invariant, this means that for the magnetic field with closed magnetic lines the invariant is not a function of pairwise linking numbers of the magnetic lines. This property is based of the following fact: an arithmetic residue of the M -invariant for a triple of closed magnetic lines, which is a model of a link with even pairwise linking numbers, coincides with the Arf-invariant (about the Arf-invariant, or, the Rokhlin-Robertello invariant, see [G-M]).

The new results concern magnetic fields on closed 3-dimensional manifolds and use the M -invariant. The manifolds with magnetic field, that we consider are not, generally speaking, simply-connected. This manifold is assumed

homogeneous and is a rational Poncaré sphere. One can try to transform results on the asymptotics and ergodicity of the M -invariant for the magnetic fields on the standard sphere S^3 to an arbitrary rational homology sphere Σ . To make this idea precise we generalize the Arf-invariants of classical semi-boundary links (including the Arf-Brown $\mathbb{Z}/8$ -invariant) (see [G-M]) and we introduce a new Arf-invariant, called the hyperquaternionic Arf-invariant.

This generalization could clarify the relationship between the M -invariant and homotopy groups of spheres. It is well-known that the helicity invariant is a specification of the Hopf invariant, see [A-Kh] for details. The Hopf invariant determines the homotopy group $\pi_3(S^2)$, the stabilization of this homotopy group is denoted by Π_1 . The group Π_1 contains the only non-trivial element with the Hopf invariant one.

The Arf-invariant describes the stable homotopy group Π_2 via the geometrical approach due to L.S.Pontrjagin. The Arf-Brown invariant describes the 2-torsion of the stable homotopy group Π_3 , this result follows from V.A.Rokhlin's theorems. The hyperquaternionic Arf-invariant describes the 2-torsion of the stable homotopy group Π_7 . This group was calculated by J.P.Serre using the algebraic approach. The complexity of the M -invariant relates with the fundamental group of rational homology sphere Σ .

The results were presented at the A.B.Sossinsky Topological Seminar in IMU September-October 2004. A preliminary result was presented at the conference on differential equations, organized by V.P.Leksin in Kolomna, June 2014. The author was supported in part by RFBR grants 15-02-01407, 15-01-06302.

The mean magnetic field equation

Let us consider, as in [R], the domain Ω in \mathbb{R}^3 , which is compact for simplicity, with a conductive liquid. In Ω a velocity field \mathbf{u} of the liquid and a magnetic field \mathbf{B} are well-defined. Moreover, the following decomposition of the considered vector-fields into a mean part and a random part is well defined:

$$\mathbf{B} = \bar{\mathbf{B}} + \mathbf{B}'; \quad \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'.$$

Assume that the mean velocity field $\bar{\mathbf{u}}(t)$ is done, then the equation for the mean magnetic field is following:

$$\begin{aligned} \text{rot}(\eta \text{rot} \bar{\mathbf{B}}) - \text{rot}(\bar{\mathbf{u}} \times \bar{\mathbf{B}} + \mathbf{E}) + \frac{\partial \bar{\mathbf{B}}}{\partial t} &= 0, \\ \mathbf{E} &= \bar{\mathbf{B}}' \times \mathbf{u}', \quad \text{div}(\bar{\mathbf{B}}) = 0. \end{aligned} \tag{1}$$

The equation (1) is called the kinematic dynamo equation. Assuming $\eta = 0$, $\mathbf{E} = 0$ this equation means that the magnetic field is frozen-in.

Assume that the following equation is satisfied:

$$\mathbf{E} = \alpha \bar{\mathbf{B}} - \beta \text{rot}(\bar{\mathbf{B}}). \quad (2)$$

Then, using the condition that α changes the sign with respect to the mirror symmetry and using additional simplifying assumptions we get:

$$\alpha \sim \overline{(\mathbf{u}', \text{rot}(\mathbf{u}'))}, \quad (3)$$

where the function $(\mathbf{u}', \text{rot}(\mathbf{u}'))$ is called the density of (a small-scaled) the hydrodynamic helicity. Denote the hydrodynamic helicity by $\chi_{\mathbf{u}'} = \int (\mathbf{u}', \text{rot}(\mathbf{u}')) d\Omega$.

Take the scalar product of the both sides of the equation (2) with the vector $\bar{\mathbf{B}}$, assuming for simplicity that $\eta = 0$, and take the integral over the domain Ω . We get, using $\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t}$, the equation, which describes the transport of the magnetic helicity $\chi_{\bar{\mathbf{B}}} = \int (\mathbf{A}, \bar{\mathbf{B}}) d\Omega$:

$$\frac{d\chi_{\bar{\mathbf{B}}}}{dt} = 2\alpha \int (\bar{\mathbf{B}}, \bar{\mathbf{B}}) d\Omega - 2\beta \int (\bar{\mathbf{B}}, \text{rot}(\bar{\mathbf{B}})) d\Omega. \quad (4)$$

The integral $U_{\bar{\mathbf{B}}} = 2 \int (\bar{\mathbf{B}}, \bar{\mathbf{B}}) d\Omega$ is called the magnetic energy (of the mean field), the integral $\chi_{\text{rot}\bar{\mathbf{B}}} = 2 \int (\bar{\mathbf{B}}, \text{rot}(\bar{\mathbf{B}})) d\Omega$ is called the current helicity (of the mean field).

Topological considerations concerning the transport equation of the magnetic helicity

In the paper [M] by K.Moffatt the equation (4) is discussed from point of view of geometry of magnetic lines. Assume that a support of a magnetic field consists of a finite set of magnetic tubes, see [B-F]. This means that the magnetic fields $\bar{\mathbf{B}}, \mathbf{B}'$ are inside the tubes and is tangent to the surfaces of the tubes. Additionally, assume that the same collection of the tubes is a support of a velocity field \mathbf{u}' . With this assumption the vorticity field points along the central axis of the each tube.

The magnetic and hydrodynamic tubes one may define such that the following condition, which is called "force-free" condition is satisfied: $\text{rot}\mathbf{u}' \sim \mathbf{u}', \text{rot}\mathbf{B}' \sim \mathbf{B}'$.

With the considered assumption it is not hard to prove, using the formula (3), that the mean magnetic field $\bar{\mathbf{B}}$ in the collection of tubes tends $|\alpha|$ -exponentially to $+\infty$, if the absolute value of α is sufficiently large. This is called the α -effect.

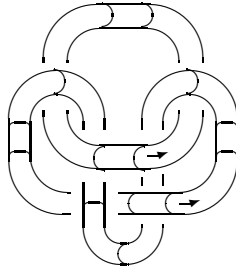
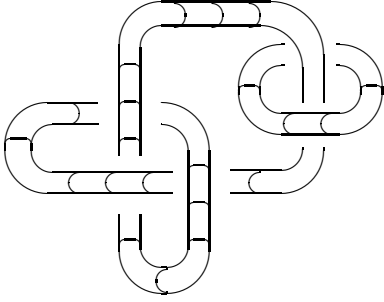
The contribution of the second term in the right side of the equation (4) is given by the Calugareanu formula, see [M-R]. The magnetic helicity inside the only magnetic tube is calculated by the formula:

$$\chi_{\mathbf{B}} = \Phi^2 Lk,$$

where Φ is the integral magnetic flow through a transversal section of the tube, Lk is the self-linking number of the magnetic tube. For two magnetic tubes Ω_1, Ω_2 with magnetic flows Φ_1, Φ_2 the magnetic helicity is calculated by the formula:

$$\chi_{\mathbf{B}} = \Phi_1^2 Lk(1, 1) + 2\Phi_1\Phi_2 Lk(1, 2) + \Phi_2^2 Lk(2, 2).$$

Magnetic tubes



Assume that magnetic lines in each magnetic tube on the figure are untwisted with respect to the plane of the projection. Then for the first pair of magnetic tubes we have: $Lk(1, 1) = -3$, $Lk(1, 2) = \pm 1$, $Lk(2, 2) = 0$; for the second pair of magnetic tubes we have: $Lk(1, 1) = -2$, $Lk(1, 2) = Lk(2, 2) = 0$.

Assume that a magnetic field is inside the only magnetic tube. In this case the self-helicity of the magnetic tube Ω is calculated by the Calugareanu formula:

$$Lk = Wr + Tw.$$

(for the simplicity we assume that the magnetic flow through the magnetic tube is equal to 1). Consider the central line l of the magnetic tube Ω and consider the vector field ξ , which is perpendicular to l , and points from l to anyone magnetic line in U . Let us say that the magnetic tube Ω is "hairless" if the derivative of the field along the tangent vector of l is trivial at each point of l .

For an arbitrary magnetic tube Ω with a central line l consider the "hairless" magnetic tube Ω_0 with the same central line and the same (unite) magnetic flow. The self-linking number for Ω_0 is calculated by the Calugareanu formula as the sum of the self-linking number Wr of Ω_0 with the twisting number Tw of the magnetic tube Ω with respect to Ω_0 .

It is not hard to prove the following equation:

$$\chi_{\text{rot}\mathbf{B}} = C\Phi^2 Tw,$$

where $\chi_{\text{rot}\mathbf{B}}$ is the current helicity of the magnetic field in Ω , Tw is the twisting number of the magnetic tube Ω , C is the coefficient of an order 1, which depends of a geometry of magnetic lines inside U . Therefore the second term in the formula (4) detects an untwist of the magnetic tube, see [A-K-K] for more details.

Quadratic helicity and ergodic integrals

Assume for a simplicity that magnetic lines in a magnetic tube Ω are closed. The magnetic tube Ω is characterized by the magnetic helicity integral, this integral is equal to the mean pairwise linking number of magnetic lines in the magnetic tube Ω , which is normalized by magnetic flows through the collection of infinitesimal magnetic lines Ω .

The magnetic tube Ω is also characterized by various combinatorial invariants $I(L_1, L_2, L_3)$, which are calculated for various collections $\{L_1, L_2, L_3\}$ of k magnetic lines (we assume $k = 3$ for simplicity). In this case we may

assume that lines of collections are inside the magnetic tubes $\Omega_1, \Omega_2, \Omega_3$ correspondingly, some magnetic tubes could coincide. In a particular interesting case we have the only magnetic tube, magnetic lines of the collections are inside of this tube.

What are required conditions for a combinatorial invariant I , which can be apply to describe magnetic fields? From the consideration above of the equation (4) we have to assume the following conditions:

- C1. The invariant I is of a finite-type invariant of an order t in the sense of V.A.Vassiliev.

- C2. The invariant I is characterized by a positive integer s , which is called the asymptotic denominator. Take a link (L_1, L_2, L_3) , which is formed by central lines of disjoint magnetic tubes $\Omega_1, \Omega_2, \Omega_3$. Denote by (rL_1, rL_2, rL_3) the r -time spinning link, which is constructed from (L_1, L_2, L_3) by the r -fold spinning along the central line of the corresponding magnetic tubes $\Omega_1, \Omega_2, \Omega_3$. The following equation is satisfied:

$$r^{3s} I(L_1, L_2, L_3) = I(rL_1, rL_2, rL_3) + O(r^{3s-1}).$$

- C3. Assume we have two disjoint magnetic tubes Ω_2, Ω_3 and we have two parallel magnetic lines, which is a 2-component link (L_1, L_2) in Ω_2 , and a magnetic line L_3 is a central line in Ω_3 . Take a magnetic tube Ω_2^{tw} , which is obtained from the magnetic tube Ω_2 by a twist, $\Omega_2 \mapsto \Omega_2^{tw}$, $Tw(\Omega_2) = Tw(\Omega_2^{tw}) + const$. Take two parallel magnetic lines L_1^{tw}, L_2^{tw} in Ω_2^{tw} . Take the r -time spinning link (rL_1, rL_2) , each component of this link is rotated along the central line $L_1 = L_2$ of Ω_2 r times. Take the r -time spinning link (rL_1^{tw}, rL_2^{tw}) , each component of this link is rotated along the central line $L_1^{tw} = L_2^{tw}$ of Ω_2^{tw} r times. Take 3-component links (rL_1, rL_2, rL_3) , $(rL_1^{tw}, rL_2^{tw}, rL_3)$. The following formula is satisfied:

$$I(rL_1, rL_2, rL_3) - I(rL_1^{tw}, rL_2^{tw}, rL_3) = O(r^{3s}).$$

- C4 (Condition-Definition). Assume that the invariant I is not a function of pairwise linking numbers of components of the link (for a 3-component link we get 3 pairwise linking numbers). In this case we say that I is a higher invariant.

Quadratic helicity

We shell give an non-formal definition. A precise definition of the quadratic helicity (without the assumption that a magnetic line is non-closed) is presented in [A]. Consider a finite collection $\{l_i\}$ of N magnetic lines (each line is closed for simplicity). $1 \leq i \leq N$, $N \gg 1$, and consider a symmetric

$N \times N$ matrix with zero elements on the main diagonal, which is defined by the pairwise linking numbers $lk(i, j)$, $1 \leq i < j \leq N$ of magnetic lines.

Define the quadratic helicity integral $\chi^{(2)}$ by the formula:

$$\chi^{(2)} = \Phi_i \Phi_j \Phi_k \sum_{i,j,k} lk(i, j) lk(j, k), \quad i \neq j, j \neq k, k \neq i, \quad 1 < i, j, k < N, \quad (5)$$

where Φ_i, Φ_j, Φ_k are integral magnetic flows through infinitesimal thin magnetic tubes $\Omega_i, \Omega_j, \Omega_k$, which are formed by the corresponding magnetic lines.

The quadratic helicity satisfy Conditions C1, C2, C3 for $s = \frac{4}{3}, t = 2$. It is interesting to remark that

$$t = \frac{3s}{2}. \quad (6)$$

For a higher invariant we get $t < \frac{3s}{2}$.

The relative quadratic helicity is well-defined. In a particular case the relative quadratic helicity express the Total invariant of magnetic braids, introduced in [Y-H].

An ergodic integral

Let us recall an approach by V.I. Arnol'd toward a description of the magnetic helicity integral as an asymptotic ergodic Hopf invariant of magnetic lines, see [A-Kh]. Assume that the magnetic field \mathbf{B} is represented by a finite collection of magnetic tubes $\Omega \subset \mathbb{R}^3$. Denote by $F(t) : \Omega \rightarrow \Omega$ the ergodic magnetic flow along magnetic lines. By the Birkhoff Theorem the function (\mathbf{A}, \mathbf{B}) admits the ergodic average, denote this function by $f : \Omega \rightarrow \mathbb{R}$. The function f is invariant with respect to the magnetic flow F . The function F is constant restricted to each magnetic line - a trajectory of the flow F . The function f is called the helicity density.

The function f is frozen-in with respect to volume-preserved transformations of \mathbb{R}^3 . The function f is integrable and, moreover, an arbitrary positive power $f^k = f \cdot \dots \cdot f$, $k \in \mathbb{Z}, k \geq 1$ is integrable. The magnetic helicity integral is the result of the integration of f over Ω . The quadratic magnetic helicity integral is the result of the integration f^2 over Ω .

The ergodic theorem allows us to transform a combinatorial invariant of closed magnetic lines into asymptotic invariants of magnetic fields with non-closed magnetic lines, details of the construction are in [A]. Invariants of magnetic fields are given by ergodic integrals.

A higher invariant of magnetic lines

The M -Invariant for a triple of magnetic tubes

Consider a magnetic field $\mathbf{B} = \cup_i \mathbf{B}_i$ with a support into 3 magnetic tubes Ω_i , $i = 1, 2, 3$ correspondingly. Assume that inside the each magnetic tube a coordinate system $U_i \cong D^2 \times S^1$ is fixed. Assume that this coordinate system corresponds with the standard volume form in \mathbb{R}^3 and the magnetic field \mathbf{B}_i points strictly along the S^1 -coordinate of the system. This assumption simplifies calculations and gives no loss of a generality.

The integral magnetic flow of \mathbf{B}_i trough the cross-section of the magnetic tube U_i is denoted by Φ_i . The integral linking number $\int_{U_i} (\mathbf{A}_j, \mathbf{B}_i) dx = \Phi_i \Phi_j lk(i, j)$ of magnetic tubes U_i, U_j is denoted by (i, j) , $i, j = 1, 2, 3$ $i \neq j$.

A multivalued function with the period (i, j) , which is a restriction of the scalar branch of the vector-potential \mathbf{A}_j on the magnetic tube U_i denote by $\varphi_{j,i} : U_i \rightarrow \mathbb{R}$. The function $\varphi_{j,i}$ is well-defined up to an additive constant. Consider a function

$$\phi_1 = (3, 1)\varphi_{2,1} - (1, 2)\varphi_{3,1} : U_1 \rightarrow \mathbb{R},$$

which is well-defined by means of multivalued functions $\varphi_{2,1}, \varphi_{3,1}$ up to an additive constant. To fix the constant, we assume that the following equation is satisfied: $\int_{U_1} \phi_1 dx = 0$. Define the functions ϕ_2, ϕ_3 by analogous formula.

Define the vector

$$\begin{aligned} \mathbf{F} = & (1, 3)(2, 3)\mathbf{A}_1 \times \mathbf{A}_2 + (2, 1)(3, 1)\mathbf{A}_2 \times \mathbf{A}_3 + (3, 2)(1, 2)\mathbf{A}_3 \times \mathbf{A}_1 \\ & - \phi_1 \mathbf{B}_1(2, 3) - \phi_2 \mathbf{B}_2(3, 1) - \phi_3 \mathbf{B}_3(1, 2). \end{aligned}$$

Obviously, the equation $\text{div}(\mathbf{F}) = 0$ is satisfied.

The vector-potential \mathbf{G} , $\text{rot} \mathbf{G} = \mathbf{F}$ and the integral $\int_{\mathbb{R}^3} (\mathbf{G}, \mathbf{F}) dx$ are well-defined. This integral is modified into the required invariant of volume-preserved diffeomorphisms. This modification includes the following extra 10 terms:

$$e_{1,2,3} = -2(1, 2)(2, 3)(3, 1) \left(\int_{\mathbb{R}^3} \langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle dx \right)^2, \quad (7)$$

$$f_1 = -2 \left(\int_{U_1} \varphi_{2,1}^{var} (\mathbf{grad} \varphi_{3,1}^{var}, \mathbf{B}_1) dU_1 \right) \left(\int_{\mathbb{R}^3} \langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle dx \right), \quad (8)$$

$$d_{1,1} = -(2, 3)^2 \int \phi_1^2 (\mathbf{A}_1, \mathbf{B}_1) dU_1, \quad (9)$$

$$d_{1;3} = (2, 3)(1, 2) \int \phi_1^2(\mathbf{A}_3, \mathbf{B}_1) dU_1. \quad (10)$$

In the formula (8) the terms $\varphi_{3,1}^{var}$, $\varphi_{2,1}^{var}$ are defined from $\varphi_{3,1}$, $\varphi_{2,1}$ correspondingly, see [A2]. The extra 6 terms are defined by cyclic permutation of the indexes $\{1, 2, 3\}$ in the formulas (8), (9), (10).

In [A2] the following result is proved.

Theorem 1. *The integral expression*

$$M(\mathbf{B}) = \int_{\mathbb{R}^3} (\mathbf{G}, \mathbf{F}) dx + e_{1,2,3} + \sum_{i=1,2,3} f_i + d_{i,i} + d_{i,i+2} \quad (11)$$

is invariant with respect to volume-preserved diffeomorphisms.

The invariant M of 3 closed magnetic lines

Assume that $\Phi_1 = \Phi_2 = \Phi_3 = 1$ and take a limit in the formula (4), when the thickness of magnetic tubes tends to zero. The result satisfies the following definition.

Definition

For an arbitrary 3-component link $\mathbf{L} \subset \mathbb{R}^3$ define a space (non-connected) $Conf^r(\mathbf{L}) = (\mathbf{L})^r$ as the Cartesian product of r copies of \mathbf{L} . The space $Conf^r(\mathbf{L})$ is called the configuration space of the link \mathbf{L} .

Let

$$F : Conf^r(\mathbf{L}) \rightarrow \mathbb{R} \quad (12)$$

be an arbitrary integrable function on the configuration space.

Theorem 2. *Each term in the expression (4) is defined by the integral of a corresponding function on the configuration space of the link \mathbf{L} .*

Let us formulate an analogous definition for a function on the configuration space of magnetic lines.

Ergodic integrals and quasi-ergodic integrals.

Let \mathbf{B} , $\operatorname{div}(\mathbf{B}) = 0$ be a smooth magnetic field in \mathbb{R}^3 with a support inside a finite collection of magnetic tubes $\Omega \subset \mathbb{R}^3$, the magnetic field \mathbf{B} is tangent to the surface boundary of Ω and non-vanishes inside Ω .

Define the configuration space $K_{q,r}$ of magnetic lines, where p, r are given positive integers. For an arbitrary $T > 0$ (sufficiently great) consider collections of r magnetic lines L_1, \dots, L_r of \mathbf{B} , the each line of the collection is parametrized by the standard segment $[0, T]$ and is issued from points $\{l_1, \dots, l_r\}$ in Ω correspondingly. Define collections of $r(q+1)$ points, this collection consists of r subcollections, $q+1$ points in each subcollection. The first subcollection $\{l_1; x_1, \dots, x_q\}$ consists of $q+1$ points, including the initial point l_1 on the magnetic line L_1 . The second subcollection $\{l_2; x_{q+1}, \dots, x_{2q}\}$ consists of $q+1$ points, each point is on the second magnetic line L_2 , etc., the subcollection $\{l_r; x_{q(r-1)+1}, \dots, x_{qr}\}$ consists of $q+1$ points, the each point belongs to the corresponding magnetic line L_r . Obviously, the each point x_{qj+i} is well-defined by the corresponding parameter t_{qj+i} , $1 \leq j \leq r$, $1 \leq i \leq q-1$, $0 \leq t_{qj+i} \leq T$ of the magnetic flow, which transport the point l_j to the point x_{qj+i} along the magnetic line L_j .

Let us say that the function $F : K_{q,r} \rightarrow \mathbb{R}$ determines an ergodic integral, if the following conditions are satisfied:

- -1. For almost an arbitrary point $\{l_1, \dots, l_r\} \in U^r$ the mean value $\bar{F} : K_{q,r} \rightarrow \mathbb{R}$ (in the sense of Cesàro) of the function F with respect to position of points $\{x_1, \dots, x_q, \dots, x_{q(r-1)+1}, \dots, x_{qr}\}$ is well-defined. By definition \bar{F} is induced by a function in the domain U^r with respect to the projection $\pi : K_{q,r} \rightarrow U^r$, $\pi(z) = \{l_1, \dots, l_r\}$, $z \in K_{q,r}$; denote this function by $\bar{F} : U^r \rightarrow \mathbb{R}$.

- -2. The function $\bar{F} : U^r \rightarrow \mathbb{R}$ is locally integrable and is integrable.

The ergodic integral $I(\mathbf{B})$ is defined as the integral of the function \bar{F} over the domain U^r .

Let us say that a function $F : K_{q,r} \rightarrow \mathbb{R}$ determines a quasi-ergodic integral, if a linear mapping $X : K_{q,r} \rightarrow \mathbb{R}$ with respect to variables $\{x_1, \dots, x_q, \dots, x_{q(r-1)+1}, \dots, x_{qr}\}$ is well-defined, and, moreover, for an arbitrary $p \in \mathbb{R}$ the restriction of F to $X^{-1}(p) \subset K_{q,r}$ satisfy Conditions -1, -2; moreover, for an arbitrary $p > 0$ the integral $f(p) = \int \bar{F} d(X^{-1}(p))$ determines an absolute bounded function $f(p) : \mathbb{R}_+ \rightarrow \mathbb{R}$, $p > 0$. Additionally, if magnetic lines, issued from the points $\{l_1, \dots, l_r\}$ are closed, the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is periodic.

The quasi-ergodic integral $I(\mathbf{B})$ is defined as a mean value of the function

$f(p)$ over \mathbb{R}_+ . Generally speaking, this integral is multivalued and takes the value into a segment. In the case magnetic lines of \mathbf{B} are closed, f is periodic and a value $I(\mathbf{B})$ is well-defined.

Theorem 3. *The terms $\int_{\mathbb{R}^3}(\mathbf{G}, \mathbf{F})dx$, $e_{1,2,3}$, f_i in the formula of M , presented in (2), are ergodic integrals. The terms $d_{i,i}$, $d_{i,i+2}$ in (2) are quasi-ergodic integrals.*

In the paper [A3] the following theorem is proved.

Theorem 4. *Assume that magnetic lines of \mathbf{B} inside Ω are closed. Then the invariant M satisfy Condition C1 for $t = 7$, Condition C2 for $s = 12$, and Conditions C3, C4.*

Proof of Theorem 3

A particular proof of Theorem is in [A4] (Theorem 3.1,(1) and Lemma 4.1.). I present a simplification of the proof for the main term $\int_{\mathbb{R}^3}(\mathbf{G}, \mathbf{F})dx$ with simple estimations of the integral.

Assume that a magnetic field \mathbf{B} is inside a finite collection of generic magnetic tubes, $\text{supp}(\mathbf{B}) = \Omega \subset \mathbb{R}^3$. Recall the definition of the term W of the integral W .

Coordinates of a point in $K_{3,4;2}$ are given by collections $\{l_1, t_{1,1}, \dots, t_{1,4}, l_2, t_{2,1}, \dots, t_{2,4}, l_3, t_{3,1}, \dots, t_{3,4}; y_1, y_2\}$, where $l_i \in U_i$, $t_{i,j} \in [0, T] \subset \mathbb{R}_{i,j}$, $j = 1, 2, 3, 4$, $y_1, y_2 \in \mathbb{R}^3$.

Define the evolution mapping $F : K_{3,4;2} \rightarrow \Omega_1^4 \times \Omega_2^4 \times \Omega_3^4$ by the formula

$$F(l_1, t_{1,1}, \dots, t_{1,4}, l_2, t_{2,1}, \dots, t_{2,4}, l_3, t_{3,1}, \dots, t_{3,4}) =$$

$$(g^{t_{1,1}}(l_1), \dots, g^{t_{1,4}}(l_1), g^{t_{2,1}}(l_2), \dots, g^{t_{2,4}}(l_2), g^{t_{3,1}}(l_3), \dots, g^{t_{3,4}}(l_3),$$

where g^t is the magnetic flow of \mathbf{B} . From this formula the space $K_{3,4;2}$ is the configuration space of 17-points: $3(1 + 4)$ points $\{l_i, g^{t_{i,1}}(l_i), g^{t_{i,2}}(l_i), g^{t_{i,3}}(l_i), g^{t_{i,4}}(l_i)\}$ are on the magnetic lines, which are issued from l_i , $i = 1, 2, 3$, and points $(y_1, y_2) \in (\mathbb{R}^3)^2$ are arbitrary. The standard volume form $dK_{3,4}$ on the space $K_{3,4;2}$ is well-defined.

The first step of the construction includes a definition of a function $W_{3,4;2} : K_{3,4;2} \rightarrow \mathbb{R}$, which is called the density function. The density function is not the lift of a function on Ω^3 by the projection $\pi : K_{3,4;2} \rightarrow \Omega^3$. The mean asymptotic value of the function $W_{3,4;2}$ over the coordinates $t_{i,j}$, which is well-defined almost everywhere, depends of the parameters $(l_1, l_2, l_3; y_1, y_2)$. The last second step of the construction is a construction of a limiting tensor, this proves that the integral of $W_{3,4;2}$ over $\Omega^3 \times (\mathbb{R}^3)^2$ is well-defined.

Let us use the Gauss integral to calculate W in the following formulas:

$$(2, 3)(3, 1)^2(1, 2)\gamma_{t_{1,1}, t_{2,1}, t_{2,2}, t_{3,1}}(\vec{\alpha}_{1,2}(x_1, x_{2,1}; y_1), \vec{\alpha}_{2,3}(x_{2,2}, x_3; y_2)), \quad (13)$$

$$(2, 3)^2(1, 2)^2\gamma_{t_{1,1}, t_{1,2}, t_{2,1}, t_{2,2}}(\vec{\alpha}_{1,2}(x_{1,1}, x_{2,1}; y_1), \vec{\alpha}_{1,2}(x_{1,2}, x_{2,2}; y_2)). \quad (14)$$

In this formula by $\gamma(\quad; \quad, \quad)$ is denoted the value of the kernel of the Gauss integral at a pair of corresponding vectors, the vectors of the pair depend of the parameters $(x_1, x_{2,1}, x_{2,2}, x_3)$ and are attached to the points y_1, y_2 correspondingly, the vectors $\vec{\alpha}_{1,2}(x_1, x_{2,1}; y_1)$, $\vec{\alpha}_{2,3}(x_{2,2}, x_3; y_2)$ in (13) (for (14) the formulas are similar) are given by (17), (18). The terms (13), (14) are well-defined in the asymptotic limit of all the positions of the points $x_{i,j}$. For short we take $x_1 = g^{t_{1,1}}(l_1)$, $x_3 = g^{t_{3,1}}(l_3)$. The integration over the variables y_1, y_2 is taken after the asymptotic limit.

Let us investigate the term (13) only, for the term (14) the proof is analogous. The coordinates $\{t_{1,1}, \dots, t_{1,4}, t_{2,1}, \dots, t_{2,4}, t_{3,1}, \dots, t_{3,4}\}$ are divided into the following 2 groups of coordinates, the coordinates of the first group $\{t_{1,1}, t_{2,1}, t_{2,2}, t_{3,1}\}$ are re-denoted by $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3\}$ correspondingly. The coordinates of the second group $\{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,2}, t_{3,3}, t_{3,4}\}$ are re-order as following: $\{t_{1,2}, t_{3,2}, t_{1,4}, t_{3,4}, t_{1,4}, t_{2,3}, t_{2,3}, t_{3,4}\}$ and are re-denoted by $\{\rho_{1,1}, \rho_{3,1}, \rho_{1,2}, \rho_{3,2}, \rho_{1,3}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\}$ correspondingly.

Let us define the factors in the formula (13). Using the 4 points of the first group $x_1 = g^{\tau_1}(l_1)$, $x_{2,1} = g^{\tau_{2,1}}(l_2)$, $x_{2,2} = g^{\tau_{2,2}}(l_2)$, $x_3 = g^{\tau_3}(l_3)$, define the integral kernel

$$\gamma_{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3}(\vec{\alpha}_{1,2}(x_1, x_{2,1}), \vec{\alpha}_{2,3}(x_{2,2}, x_3); y_1, y_2). \quad (15)$$

Using the last 8 points of the second group $g^{\rho_{1,1}} = z_{1,1}$, $g^{\rho_{3,1}} = z_{3,1}$, $g^{\rho_{1,2}} = z_{1,2}$, $g^{\rho_{3,2}} = z_{3,2}$, $g^{\rho_{1,3}} = z_{1,3}$, $g^{\rho_{2,3}} = z_{2,3}$, $g^{\rho_{2,4}} = z_{2,4}$, $g^{\rho_{3,4}} = z_{3,4}$, define the integral kernel to calculate $(2, 3)(3, 1)^2(1, 2)$ by obvious way, see [A-Kh] for the integral formula of the linking number. The product of the expressions gives (13).

Let us prove that for almost arbitrary collection $(l_1, l_2, l_3; y_1, y_2)$ there exists the asymptotic mean value of the expression (15) with respect to the variables $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3\}$. Denote this asymptotic mean value by

$$\bar{\gamma}(l_1, l_2, l_3; y_1, y_2) \quad (16)$$

The absolute value of coordinates of the vector-potential $\mathbf{A}(x_i; y)$ at an arbitrary point $x_i \in L$ is integrable with respect to the parameter $y \in \mathbb{R}^3$. This vector-potential determines the vector-functions

$$\vec{\alpha}_{1,2}(x_1, x_{2,1}; y_1) = \mathbf{A}(x_1; y_1) \times \mathbf{A}(x_{2,1}; y_1), \quad (17)$$

$$\vec{\alpha}_{2,3}(x_3, x_{2,2}; y_2) = \mathbf{A}(x_3; y_2) \times \mathbf{A}(x_{2,2}; y_2). \quad (18)$$

This vector-functions for arbitrary fixed y_1, y_2 are integrable with respect to the parameters $\{x_1, x_{2,1}, x_{2,2}, x_3\}$.

By the Birkhoff Theorem the vector-functions $\vec{\alpha}_{1,2}, \vec{\alpha}_{2,3}$ in (15) admit the asymptotic limits with respect to the first group coordinates. The mean vector-functions are denoted by $\vec{\bar{\alpha}}_{1,2}(l_1, l_2)(y_1), \vec{\bar{\alpha}}_{2,3}(l_2, l_3)(y_2)$, this vector-functions depend formally of the points (l_1, l_2, l_3) , but, in fact, depend of the triple of magnetic lines L_1, L_2, L_3 only.

The integral kernel (15) is calculated algebraically and the term (16) is well-defined for almost arbitrary $(l_1, l_2, l_3; y_1, y_2)$. Analogously, the integral kernel $W_{3,4,2}$, corresponded to (13), admits the mean value over all the variables $\{\tau_1, \tau_{2,1}, \tau_{2,2}, \tau_3; \rho_{1,1}, \rho_{3,1}, \rho_{1,2}, \rho_{3,2}, \rho_{1,3}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\}$. Denote this mean value by

$$\bar{W}(l_1, l_2, l_3; y_1, y_2). \quad (19)$$

The product of pairwise asymptotic linking numbers is well-defined for almost arbitrary collections of pairs of magnetic lines $(l_1, l_2), (l_2, l_3), (l_3, l_1)$ (see [A]). The vector (19) is well-defined and the first step of the construction is described.

Pass to the second step of the construction and prove that the integrals (16), (19) over $\Omega^3 \times (\mathbb{R}^3)^2$ are well-defined. Estimate the total term (13) by a limiting tensor, which is absolutely integrable over the configuration space $\Omega^3 \times (\mathbb{R}^3)^2$. Denote by $a(x_1, x_{2,1}, x_{2,2}, x_3)$ the absolute value of the term (15) (the value $+\infty$ is admitted) after the integration over the variables y_1, y_2 . Assume firstly that the points $x_1, x_{2,1}, x_{2,2}, x_3$ belong to the triple of the segments of magnetic lines, which are pairwise close to each other. Denote by δ a small parameter, which is the distance of the segment on L_2 to the segments on $\{L_1, L_3\}$ (for short we assume that the segment on L_1 is closer that the segment on L_3 to the segment on L_2).

Lemma 1. *Let $y_1 = y_2 = x_{1,1} = x_{1,2} = x_2 = x_3$, and $\omega > 0$ be a given positive (arbitrary small) number, constants $\delta_0 > \delta$ be arbitrary. Take an arbitrary non-degenerate δ -variation of the magnetic line L_1 and an arbitrary variation of the magnetic line L_3 , which is estimated from above by δ and from below by δ_0 . Then the absolute integral value of the term*

$$a(x_1, x_{2,1}, x_{2,2}, x_3) \quad (20)$$

over arbitrary ε -variations of points $y_1, y_2, x_1, x_{2,2}, x_3$ (the point $x_{1,1}$ is fixed) along the corresponding segments of magnetic lines is estimated by $C\delta^{-1-\omega}$, where the positive constant C depends only on ε . The constant ε depends on the norm of the 2-jets of \mathbf{B} in Ω and depends no of δ .

Remark

By the results of [A4], one may replace $C\delta^{-1-\omega}$ by $C\log(\delta^{-1})$ in the lemma.

Proof of Lemma 1

To simplify the notation put $\varepsilon = 1$. The singularity in the configuration space is of the order r^{-10} , where r is the distance in \mathbb{R}^3 which corresponds to the parameter of deformation. This formal order includes the order -2 of the each magnetic dipole (4 dipoles), the order of the kernel in the Gauss integral, given by $\text{dist}(y_1, y_2)^{-2}$. The integration of the term (20) is over the 6-dimensional domain of the variables y_1, y_2 and of a 3-dimensional domain, of the variables $x_{1,1}, x_{1,2}, x_2, x_3$. As the result, we get that the singularity of (20) is of the formal order -1 .

After the deformation, described in the lemma, the term (20) is well-defined and integrable. To calculate this generic term, we integrate singular functions of the order r^{-6} (the coordinate r is the distance between the parameters $x_{1,1}, x_{1,2}$ on the line L_1) over 7-dimensional space. The integral is well-defined. A formal estimation (over the parameter δ) of the deformation of the singularity proves Lemma. 1.

Let us estimate W by absolute value using the lemma. Consider the cube with the edge of the length T in the configuration space, which is given by the parameter of the magnetic flow. The configuration space is a union of a finite number of small cubes. Let us define a limiting tensor of $W_{3,4,2}$ in each cube. Recall that the limiting tensor is absolutely integrable over the configuration space. and estimates the absolute value of $W_{3,4,2}$.

We start with cubes, called diagonal cubes, which are closed to top singularities, which are described in Lemma 1, up to parallel translations of all 4-points along the magnetic flow. The last cubes in the configuration space, called peripheral cubes, are defined analogously.

In each diagonal cube we get the estimation from Lemma 1. In an arbitrary peripheral cube estimations is more simple, and formally are given by the same formulas, δ_0 is not a small parameter. As the result we get that the expression (15) is estimated by a function of the order $\delta^{-1-\omega}$, where δ is the minimal pairwise distance between segments of magnetic lines (if there is a pair of close segments of magnetic line) and by a function of the order 1, if all the segments are pairwise non-closed.

By the Holder inequality we get:

$$\int f g dx \leq \left(\int |f|^q dx \right)^{\frac{1}{q}} \left(\int |g|^p dx \right)^{\frac{1}{p}}, \quad 1 < p < 2, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In this inequality f is the limiting tensor for (15), g is the limiting tensor with logarithmic singularities for the term $(2, 3)(3, 1)^2(1, 2)$, which is much simple. We use the denominator $p = 1 + \omega$ and a large denominator q . The function (19) is integrable and the main term W is given by an ergodic integral.

Examples of Magnetic Knots in the standard sphere S^3 and in several homogeneous manifolds

In this section we consider examples of magnetic knots with closed magnetic lines (or with magnetic lines on family of surfaces) inside compact (homogeneous) manifolds, for which M -invariant is non-vanished. The Examples *I* and *II* are generalizations with non-simply connected manifolds. For this examples Theorems 1,2,4 are conjectured.

A one-parametric family of magnetic knots in S^3

Consider the standard singular fibration $S^3 \rightarrow S^2$ with 2 singular linked circles $S_1^1 \subset S^3$, $S_2^1 \subset S^3$, and with Hopf family of regular tori T_t , $t \in [1, 2]$ between this two circles, T_1, T_2 are shrined into S_1^1 and S_2^1 correspondingly. Consider the Cartesian coordinate system (x, y, z) on $S^3 \setminus \{\infty\}$. The circle S_1^1 is the unite central circle on the plane (x, y) . The circle S_2^1 is the standard vertical z -axis, $\infty \in S_2^1$, through the origin.

Define a real parameter r , $1 \leq r \leq 2$. Define a r -parameter family of magnetic knots Υ_r in S^3 . Magnetic lines of Υ_r for each $t \in]1, 2[$ are on T_t and wind 1 time along the S_1^1 -parallel of T_t and r times along the S_2^1 -meridian of T_t . For rational r , the magnetic knot Υ_r consists of closed lines. The magnetic knots Υ_{r_1} , Υ_{r_2} , in the case $r_1 \neq r_2$, are not equivalent with respect to volume-preserved diffeomorphisms of S^3 . In the case $r = 1$ we get the standard Hopf fibration with fibers along the standard Hopf mapping $h : S^3 \rightarrow S^2$.

The combinatorial formula of the invariant $M(\mathbf{L})$, in the case \mathbf{L} is a 3-component link, is well-defined up to the sum with a polynomial $P((1, 2), (2, 3), (3, 1))$, which depends on pairwise linking numbers of \mathbf{L} (see [A3]). In the case $(1, 2) = (2, 3) = (3, 1)$, to keep asymptotic properties of M , we assume that $\deg(P(k)) \leq 11$. We define $\tilde{M} = M + P$, the invariant

\dot{M} is ergodic. Moreover, without loss of generality we assume that \dot{M} is trivial on a prescribed collection of the following 2 simplest links $\mathbf{L}_{1,1,1}$, $\mathbf{L}_{2,2,2}$.

The link $\mathbf{L}_{1,1,1}$ consists of 3 magnetic lines with pairwise linking number 1, each line is a fiber of the Hopf fibration $h : S^3 \rightarrow S^2$. By the construction, $\mathbf{L}(1) = \mathbf{L}_{1,1,1}$, where $\mathbf{L}(1)$ is the link, which is defined by an arbitrary ordered magnetic lines of the magnetic knot Υ_1 .

The link $\mathbf{L}_{2,2,2}$ is defined as following. Take the symmetric triangle with the unite edges on the plane. Take 3 circles L'_1, L'_2, L'_3 of the radius $\frac{1}{2}$ around its vertexes, which are tangent to each other in the centers of edges. Then take a small 3D deformation of $(L'_1, L'_2, L'_3) \rightarrow (L_1, L_2, L_3)$ in small neighborhoods of tangent points of the pairs (L'_1, L'_2) , (L'_2, L'_3) , (L'_3, L'_1) ; as the result we assume that the pairwise linking numbers of (L_1, L_2) , (L_2, L_3) , (L_3, L_1) are equal to +2.

Denote by $\mathbf{L}(2)$ a 3-component link, which is defined by an arbitrary ordered triple of generic magnetic lines of the magnetic knot Υ_2 . It is not difficult to prove that pairwise linking numbers of $\mathbf{L}(2)$ and $\mathbf{L}_{2,2,2}$ coincide.

By the construction $\mathbf{L}_{2,2,2}$ is distinguished from $\mathbf{L}(2)$ by the commutator of 3-components (or, equivalently, by the Δ -moves of 3 components). By the following lemma and the combinatorial formula of M from [[A3](17)], the value $M(\mathbf{L}(2))$ is distinguished from $M(\mathbf{L}_{2,2,2})$ by a non-zero integer.

Lemma 2. *Let $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$ be an arbitrary 3-component link for which the pairwise linking coefficients $(1, 2), (2, 3), (3, 1)$ are even. Let $\mathbf{L}' = (L'_1 \cup L'_2 \cup L'_3)$ be the 3-component link, which is the result of a Δ -move of \mathbf{L} with 3 different components.*

The parity of the coefficients $C_2(\mathbf{L}), C_2(\mathbf{L}')$ of the Conway polynomial are distinguished, and the invariants $Arf(\mathbf{L}), Arf(\mathbf{L}')$ are distinguished.

Remark 1. *The invariant $Arf(\mathbf{L})$ is well-defined in a less restrictive case, when all the pairwise linking numbers of \mathbf{L} are odd.*

Proof of Lemma 2

For a link $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$ which satisfies the lemma, the equation $\mu_{123}^2(\mathbf{L}) \equiv C_2(\mathbf{L}) \pmod{GCD(1, 2), (2, 3), (3, 1)}$ is proved in [M], Theorem 3.5. The equation $Arf(\mathbf{L}) \equiv \mu_{123}(\mathbf{L}) \pmod{2}$ is proved using the Gauss diagrams as in [M-P]. Arf -invariant satisfy the lemma. Lemma 2 is proved.

The invariant M is ergodic, therefore $M(\mathbf{L}(r))$ is continuously changed from $M(\mathbf{L}(1)) = 0$ to $M(\mathbf{L}(2)) \neq 0$, $1 \leq r \leq 2$.

Example I of a magnetic knot in the rational homological sphere S^3/\mathbf{Q}

In the group of unit quaternions $S\mathbb{H}$ consider the subgroup of integer quaternions $\mathbf{Q} \subset S\mathbb{H}$

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}.$$

Consider the standard (right) action $\mathbf{Q} \times S^3 \rightarrow S^3$, which is well-defined because of the diffeomorphism $S\mathbb{H} \cong S^3$. Consider the 2-sheeted covering $S\mathbb{H} \rightarrow SO(3)$, the image of the subgroup $\mathbf{Q} \subset S\mathbb{H}$ is the Klein subgroup $\mathbf{K} \subset SO(3)$, $\mathbf{K} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The Klein group acts on S^2 , this action is induced by the standard projection $SO(3) \rightarrow S^2$, the action has 6 fixed points, which are the intersection points of the standard unite sphere $S^2 \subset \mathbb{R}^3$ with the coordinate axis. The elements of \mathbf{K} acts on S^2 by rotations trough the angle π with respect to the corresponding coordinate axis.

The following commutative diagram of groups

$$\begin{array}{ccc} \mathbf{Q} \times S^3 & \rightarrow & S^3/\mathbf{Q} \\ \downarrow & & \downarrow \\ \mathbf{K} \times S^2 & \rightarrow & S^2/\mathbf{K}, \end{array} \quad (21)$$

is well-defined. In this diagram horizontal maps are projections onto the orbits of the action, the left vertical mapping is the Cartesian product of the projection $\mathbf{Q} \rightarrow \mathbf{K}$ and the composition $S^3 \cong S\mathbb{H} \rightarrow SO(3) \rightarrow S^2$, which coincides with the standard Hopf fibration, the right vertical mapping is induced from the left vertical mapping by the projection onto the orbits.

The magnetic knot in S^3/\mathbf{Q} with closed magnetic lines is well-defined by fibers of the right vertical mapping in the diagram. A generic magnetic line $L \subset S^3/\mathbf{Q}$ of this magnetic knot represents an oriented cycle $[L] \in H_1(S^3/\mathbf{Q}; \mathbb{Z})$, which is not an oriented boundary, but is a non-oriented boundary. This means that the magnetic line is a boundary of a non-oriented Seifert surface. The manifold S^3/\mathbf{Q} admits a natural trivialization of the tangent bundle, $T(S^3/\mathbf{Q}) \cong 3\varepsilon$. Take a magnetic line L_1 in the integer homology class of $[L]$. From this data the Arf-Brown invariant $\Theta(L_1) \in \mathbb{Z} \pmod{8}$ is well-defined.

A generalization of the Example

The diagram (21) is included into the following diagram:

$$\begin{array}{ccc} \Sigma \times S^3 & \rightarrow & S^3/\Sigma \\ \downarrow & & \downarrow \\ \mathbb{I} \times S^2 & \rightarrow & S^2/\mathbb{I}. \end{array} \quad (22)$$

In this diagram $\mathbf{Q} \subset \Sigma$ is the Poincaré extension of the index 15 of the integer quaternions to the fundamental group of the integer homology sphere, $\mathbf{K} \subset \mathbb{I}$ is the extension of the Klein group to the icosahedron group, the lower horisontal mapping of the diagram is a free action, the bottom mapping is the semi-free action. By the Klein uniformization [K], the icosahedron group \mathbb{I} is covered by the modular group $PSL(2, \mathbb{Z})$, which acts conform on the half-plane.

Below in the diagram (24) a quadratic extension $\mathbf{Q} \subset \mathbb{N}$ is well-defined. The quadratic extension $\mathbf{Q} \subset \mathbb{N}$ is mapped into a quadratic extension $\mathbf{K} \subset \mathbf{D}$ by the projection onto the factorgroup, where \mathbf{D} is the dihedral group of the order 8.

The inclusion $\mathbf{K} \subset \mathbb{I}$ admits no extension of the quadratic extension $\mathbf{K} \subset \mathbf{D}$ of the subgroup to a quadratic extension of the group \mathbb{I} . The minimal infinite-order extension $\mathbb{I} \subset \Upsilon$ is well-defined, where Υ is a Kleinian group, which acts conform on $\hat{\mathbb{C}}$, and this action extends the action of the Fuchsian group. The group Υ is covered by a group, which acts conform on the half-plane.

Example II of magnetic knot in the rational homological sphere S^3/\mathbf{Q}

The standard Hopf fibration $h : S^3 \rightarrow S^2$, is given by the formula $\{(z_1, z_2)\}, |z_1|^2 + |z_2|^2 = 1$,

$$h : (z_1, z_2) \mapsto \frac{z_1}{z_2}.$$

The conjugated Hopf fibration $\bar{h} : S^3 \rightarrow S^2$ is given by the formula

$$\bar{h} : (z_1, z_2) \mapsto \frac{\bar{z}_1}{z_2}.$$

The following diagram

$$\begin{array}{ccc} \mathbf{Q} \times S^3 & \rightarrow & S^3/\mathbf{Q} \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 \times S^2 & \rightarrow & \mathbb{RP}^2, \end{array}$$

is well-defined, where $\mathbf{Q} \rightarrow \mathbb{Z}/2$ is the epimorphism with the generator \mathbf{i} is the kernel, $\mathbb{Z}/2 \times S^2 \rightarrow \mathbb{RP}^2$ is the projection of the antipodal involution, see [S].

Define the magnetic knot on S^3/\mathbf{Q} by the fibers of \bar{h} . An arbitrary magnetic line $L \subset S^3/\mathbf{Q}$ of the magnetic knot is not an non-oriented boundary.

The group \mathbf{Q} , which is the fundamental group of the rational homology sphere S^3/\mathbf{Q} admits a quadratic extension $\mathbf{Q} \subset \mathfrak{N}$, which is defined below by (24). By this extension the image of the generator $\mathbf{i} \in \mathbf{Q}$ in \mathfrak{N} belongs to the commutant $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{N}$. A Seifert surface for L is well-defined as a surface with a prescribed normal bundle structure (see below the definition of this structure in Theorem 5) with a control to the Eilenberg-MacLane space $K(\mathfrak{N}, 1)$. For Seifert surfaces with prescribed normal bundle structures the hyperquaternionic Arf-invariant is well-defined as an integer (mod 16).

Examples *I*, *II* of magnetic knots on S^3/\mathbf{Q} assume that asymptotic ergodic M -invariant is generalized for magnetic knots in rational homology spheres. The parity of C_2 -coefficient of the Conway polynomial for classical links in \mathbb{R}^3 corresponds to the Arf-invariant. In the next section we determine a group W , which is called the Witt group of hyperquaternionic forms. The reason to introduce the hyperquaternionic Arf-invariant is clarify by the following diagram:

$$\begin{array}{ccc}
C_2 \text{ of the Conway polynomial} & \longrightarrow & \text{Arf invariant} \\
& & \text{of classical links} \\
\downarrow & & \downarrow \\
? & \longrightarrow & \text{hyperquaternionic Arf - invariant} \\
& & \text{of links in } S^3/\mathbf{Q}.
\end{array}$$

In the diagram by $?$ is denoted a hypothetic integer-valued finite-type invariant of links in rational homological spheres, which determines asymptotic ergodic invariants.

Hyperquaternionic Arf-invariant

Arf-invariants of immersed surfaces

Consider an immersion $\varphi : M^2 \looparrowright \mathbb{R}^3$ of a closed, generally speaking, non-oriented surface into \mathbb{R}^3 . The immersion φ up to regular cobordism represents an element of the group denoted by $Imm^{sf}(2, 1)$, we use notations as in [A-E]. The Arf-Brown invariant is an isomorphism

$$\Theta : Imm^{sf}(2, 1) \rightarrow \mathbb{Z}/8.$$

Denote $Imm^{sf}(2, 1)$ by V for short (an algebraic definition of $\Theta : V \cong \mathbb{Z}/8$, using $\mathbb{Z}/4$ -quadratic forms, is in [G-M]). If M^2 is an orientable surface, the element $\Theta([\varphi])$ belongs to the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/8$. In this case the element $\frac{\Theta([\varphi])}{4} \pmod{2}$ is called the Arf-invariant of $[\varphi]$.

Let K^3 be a closed oriented 3-dimensional manifold. Assume that a trivialization of the tangent bundle $\Psi : T(K^3) \cong 3\varepsilon$ is fixed. Assume that an immersion $\varphi : M^2 \looparrowright \mathbb{R}^3$ of a closed surface is given. The immersion φ represents an element $[\varphi]$ in the group V and the Arf-Brown invariant $\Theta([\varphi])$ is well-defined.

In the case, when M^2 is a surface with a boundary, assume that each component of the immersed curve $\varphi(\partial M^2)$ has the trivial stable Hopf invariant (= an even self-linking number). In this case the Arf-Brown invariant $\Theta([\varphi])$ is well-defined.

Group \aleph of the order 16

Consider the cyclic group C_8 of the order 8, $C_8 = \{\exp(\frac{k\pi i}{4}) \mid k \in \mathbb{Z}/8\}$. Denote by $\theta : C_8 \rightarrow C_8$, $\theta : S \mapsto S^3$, $S \in C_8$ the cubing automorphism. Let us define a group \aleph of the order 16, by attaching an element T of the order 2 by the equation $TST = S^3$, see for details [C-M], Ch.1 1.8. The following short exact sequence:

$$0 \rightarrow C_8 \rightarrow \aleph \rightarrow \mathbb{Z}/2 \rightarrow 0 \quad (23)$$

is well-defined. In this sequence the left mapping is the inclusion on the subgroup, the right mapping is the projection onto. Denote by $T\mathbf{j} \in C_8$ a generator of the subgroup $C_8 \subset \aleph$; denote by $\mathbf{j} \in \aleph$ the element $T(T\mathbf{j})$; denote by $\mathbf{k} \in \aleph$ the element $T\mathbf{j}T$; denote by $-1 \in C_8 \subset \aleph$ the element $(T\mathbf{j})^4$, denote by $-\mathbf{i}$ the element $(T\mathbf{j})^2 = \mathbf{kj}$.

Define the following short exact sequence

$$0 \rightarrow \mathbf{Q} \rightarrow \aleph \rightarrow \mathbb{Z}/2 \rightarrow 0, \quad (24)$$

where \mathbf{Q} is the integer quaternions subgroup. The group \mathbf{Q} is of the order 8, this group admits the following standard corepresentation:

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\},$$

which corresponds to the notations of the generators.

Representation $\Phi : \aleph \rightarrow S\mathbb{O}(4)$

Define a $S\mathbb{O}(4)$ -representation $\Phi : \aleph \rightarrow S\mathbb{O}(4)$ by the following matrices:

$$\Phi(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \Phi(T\mathbf{j}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (25)$$

The elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are given by the following matrices:

$$\Phi(\mathbf{i}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \Phi(\mathbf{j}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \Phi(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (26)$$

The representation $\phi = \Phi|_{\mathbf{Q}} : \mathbf{Q} \rightarrow S\mathbb{O}(4)$ is equivalent to the standard representation $\mathbf{Q} \rightarrow S\mathbb{H} \subset S\mathbb{O}(4)$.

The octahedral extension $\aleph \subset \Upsilon$ of the index 3

Let us unify short exact sequences (23), (24) into the following diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & \mathbf{K} & \subset & \mathbf{D} & \rightarrow & \mathbb{Z}/2 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \rightarrow & \mathbf{Q} & \subset & \aleph & \rightarrow & \mathbb{Z}/2 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & \mathbb{Z}/2 & \cong & \mathbb{Z}/2 & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array} \quad (27)$$

In this diagram by \mathbf{D} is denoted the dihedral group of the order 8, the projection $\aleph \rightarrow \mathbf{D}$ extends the reduction $C_8 \rightarrow C_4$ of the cyclic subgroup modulo 4, $\mathbf{K} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathbf{D}$ is the Kleinian group, $\mathbf{Q} \rightarrow \mathbf{K}$ is the natural epimorphism, which is the projection onto the central quotient $\{\pm 1\} \subset \mathbf{Q}$. The group \mathbf{K} is equipped with the representation $\mathbf{K} \rightarrow S\mathbb{O}(3)$, the image of the corresponding element $[\mathbf{i}], [\mathbf{j}], [\mathbf{k}]$ is the rotation trough the angle π with respect to the axis, which is perpendicular to the coordinate plane $P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$ in \mathbb{R}^3 correspondingly.

The group \mathbf{D} is equipped with the representation $\tilde{\lambda} : \mathbf{D} \rightarrow \mathbb{O}(3)$, the element $[T] \in \mathbf{D}$, which is define as the image of the element $T \in \aleph$ by the projection $\aleph \rightarrow \mathbf{D}$, is represented by symmetry with respect to the plane, which is perpendicular to $P_{\mathbf{i}}$, along the bisector of the coordinate planes $P_{\mathbf{j}}$ and $P_{\mathbf{k}}$. The representation $\tilde{\lambda}|_{\mathbf{K}} = \lambda$ is defined such that the representation $\phi : \mathbf{Q} \rightarrow \mathbb{S}^3 \subset S\mathbb{O}(4)$ covers the representation λ by the projection $S^3 \rightarrow S\mathbb{O}(3)$.

The representation $\tilde{\lambda} : \aleph \rightarrow \mathbb{O}(4)$ is the quadratic extension of the representation λ by the standard quadratic extension $S\mathbb{O}(3) \subset \mathbb{O}(3)$.

Define the following diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z}/3 \tilde{\times} \mathbf{K} & \subset & \mathbb{Z}/3 \tilde{\times} \mathbf{D} & \rightarrow & \mathbb{Z}/2 \rightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \rightarrow & \mathbb{I} & \subset & \Upsilon & \rightarrow & \mathbb{Z}/2 \rightarrow 0
\end{array} \tag{28}$$

The group $\mathbb{Z}/3 \tilde{\times} \mathbf{K}$ is a semi-direct product of the subgroups \mathbf{D} , $\mathbb{Z}/3$ in the icosahedron group. The subgroup $\mathbb{Z}/3$ permutes the images of quaternion units $[\mathbf{i}]$, $[\mathbf{j}]$, $[\mathbf{k}]$ in \mathbf{K} . The inclusion $\mathbb{Z}/3 \tilde{\times} \mathbf{K} \subset \mathbb{I}$ into the icosahedron group is of the index 5. The group Υ is the fundamental group of the homology Poincaré sphere. The inclusion $\mathbb{Z}/3 \tilde{\times} \mathbf{D} \subset \Upsilon$ is the quadratic extension of the inclusion $\mathbb{Z}/3 \tilde{\times} \mathbf{K} \subset \mathbb{I}$.

Lemma 3. -1. Diagram (28) is well-defined and contains the diagram (27) as a subdiagram.

-2. The groups Υ , \mathbf{B} are equipped with representations $M : \Upsilon \rightarrow S\mathbb{O}(4)$, $\mu : \mathbb{Z}/3 \tilde{\times} \mathbf{D} \rightarrow \mathbb{O}(3)$, the representations M, μ extend the representations Φ , $\tilde{\lambda}$ correspondingly.

The Witt group W of hyperquaternionic forms

Define the regular cobordism group of closed surfaces, the elements of W will be called hiperquaternionic forms. Denote this group by W , from algebraic point of view, W is a Witt group of special quadratic forms.

Define an epimorphism $\alpha : \aleph \rightarrow \mathbb{Z}/2 = \{\pm 1\}$ by the following formula: $T\mathbf{j}, T \in \aleph$, $\alpha(T\mathbf{j}) = -1$, $\alpha(T) = +1$. The kernel $Ker(\alpha)$ coincides with the dihedral subgroup $\mathbf{D} \subset \aleph$.

Define an epimorphism $\beta : \aleph \rightarrow \mathbb{Z}/2 = \{\pm 1\}$ by the following formula: $\beta(T\mathbf{j}) = -1$, $\beta(T) = -1$. The kernel $Ker(\beta)$ coincides with the quaternion subgroup $\mathbf{Q} \subset \aleph$.

Over the space $B\aleph = K(\aleph, 1)$ the canonical vector $S\mathbb{O}(4)$ -bundle is well-defined, the structure group of the canonical bundle is defined by the representation $\Phi : \aleph \rightarrow S\mathbb{O}(4)$, denote this universal bundle by A . Denote by γ the line canonical bundle over $B\mathbb{Z}/2 \cong \mathbb{P}^\infty \cong K(\mathbb{Z}/2, 1)$. Denote by $\alpha : K(\aleph, 1) \rightarrow K(\mathbb{Z}/2, 1)$ the mapping of the classifying spaces, which is associated with the homomorphism α , denote by $\beta : K(\aleph, 1) \rightarrow K(\mathbb{Z}/2, 1)$ the mapping, which is associated with the homomorphism β .

A triple (M^2, η_M, Ξ_M) is called a hyperquaternionic form, where

- M^2 is a closed, generally speaking, non-orientable surface;

- $\eta_M =: M^2 \rightarrow K(\mathbb{N}, 1)$ is a characteristic class, the composition $\alpha \circ \eta_M$ is denoted by $\eta_{\alpha;M} : M^2 \rightarrow K(\mathbb{Z}/2, 1)$, the composition $\beta \circ \eta_M$ is denoted by $\eta_{\beta;M} : M^2 \rightarrow K(\mathbb{Z}/2, 1)$;
- Ξ_M is the isomorphism $T(M) \oplus \eta_{\alpha;M}(\gamma) \oplus \eta_M^*(A) \oplus 3\eta_{\beta;M}(\gamma) \cong 10\varepsilon$, where by ε is the trivial line bundle.

In particular, by definition of Ξ_M , the characteristic class $\eta_{\alpha;M} + \eta_{\beta;M} : M^2 \rightarrow K(\mathbb{Z}/2, 1)$ corresponds to the orientation homomorphism $H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$, (denote $\eta_{\alpha;M} + \eta_{\beta;M} = \kappa_M : M^2 \rightarrow K(\mathbb{Z}/2, 1)$, this characteristic class coincides with the characteristic Stiefel-Whitney class $w_1(M)$).

On a set of all hyperquaternionic form an additive operation by a disjoint union is well defined. The standard regular cobordism relation determines an equivalence relation of quadratic hyperquaternionic forms. The cobordism group up to this equivalence relation is denoted by W , this is the required Witt group.

Definition 1. A hyperquaternionic form (M^2, η_M, Ξ_M) , for which the characteristic mapping η takes values in the subspace $K(\mathbb{Q}, 1) \subset K(\mathbb{N}, 1)$, is called a quaternionic form.

Theorem 5. The group W contains a cyclic subgroup $P \subset W$ of the order 16, $P \cong \mathbb{Z}/16$.

Definition 2. Define a subgroup $W_{\mathbb{Q}} \subset W$ in the Witt group as the group, which is generated by quaternionic forms. Define the group $W_{\mathbb{Q}}^{\odot}$, which is called the Witt group of quaternion forms. The group $W_{\mathbb{Q}}^{\odot}$ is generated by quaternion forms, the regular cobordism relation for this group assumes the following additional property:

- the structure mapping on a cobordism manifold admits a prescribed reduction to a mapping with the image in the quaternion classifying subspace $K(\mathbb{Q}, 1) \subset K(\mathbb{N}, 1)$.

By the construction, the canonical projection $p : W_{\mathbb{Q}}^{\odot} \rightarrow W_{\mathbb{Q}}$ is well-defined.

The Arf-Brown homomorphism the group $W_{\mathbb{Q}}$ onto the Witt group of $\mathbb{Z}/4$ -quadratic forms

Denote by V the Witt group of $\mathbb{Z}/4$ -quadratic forms with Arf-Brown invariants. This group is related with the Rokhlin's Signature Theorem, see [G-M]. The group V is the cyclic group of the order 8. Define the forgetful homomorphism

$$\rho^{\odot} : W_{\mathbb{Q}}^{\odot} \rightarrow V$$

from the Witt group of quaternionic forms into the Witt group of quadratic $\mathbb{Z}/4$ -forms as following.

Let (M^2, η, Ξ) be a quaternionic form represented an element in $W_{\mathbf{Q}}^{\odot}$. Consider the standard 3-skeleton $S^3/\mathbf{Q} \subset K(\mathbf{Q}, 1)$, which is represented by the standard quaternion lens space. The pull-back of the bundle A over $K(\mathbb{N}, 1)$ with respect to the inclusion $S^3/\mathbb{I} \subset K(\mathbb{I}, 1) \rightarrow K(\mathbb{N}, 1)$ is denoted by $A_{S^3/\mathbf{Q}}$. The canonical isomorphism $A_{S^3/\mathbf{Q}} \cong 4\varepsilon$ of the vector bundles over S^3/\mathbf{Q} is well-defined. The pull-back isomorphism $\eta^*(A_{S^3/\mathbf{Q}}) \cong 4\varepsilon$, determines the isomorphism $\Xi_M : \nu_M \rightarrow 7\varepsilon \oplus \kappa$, where ν_M is the stable normal bundle over M^2 . Define $\rho^{\odot}([(M^2, \eta_M, \Xi_M)]) \in V$ by the formula: $\rho^{\odot}(M^2, \eta_M, \Xi_M) = (M^2, \Xi_M), [(M^2, \Xi_M)] \in V$.

Lemma 4. *The homomorphism $\rho^{\odot} : W_{\mathbf{Q}}^{\odot} \rightarrow V$ is decomposed as following:*

$$\rho^{\odot} = \rho \circ p : W_{\mathbf{Q}}^{\odot} \rightarrow W_{\mathbf{Q}} \rightarrow V,$$

where the homomorphism $p : W_{\mathbf{Q}} \rightarrow V$ is well-defined and is an epimorphism onto the index 2 subgroup in V of elements of the order 4.

Proof of Lemma 4

Consider the standard transfer homomorphism with respect to the subgroup $\mathbf{Q} \subset \mathbb{N}$, denote the transfer homomorphism by $! : W \rightarrow W_{\mathbf{Q}}^{\odot}$. The following lemma is required.

Lemma 5. *The image of the transfer homomorphism $! : W \rightarrow W_{\mathbf{Q}}^{\odot}$ is inside the kernel $\text{Ker} \rho^{\odot}$.*

Proof of Lemma 5

A given arbitrary hyperquaternionic form (M^2, η_M, Ξ_M) , is represented by a connected surface. Take a geometrical stabilization of the surface M^2 by a connected sum with 2 mirror copies of Moebius bands, the generators of the bands are represented by the element T (we say that a band of the considered type is a T -band). Denote the result of the stabilization again by (M^2, η_M, Ξ_M) . As the result, the surface M^2 is a connected sum of Moebius bands, which are represented by the elements \mathbf{j} , or by \mathbf{k} (we say that a band of the considered type is a quaternion band).

Take the decomposition of M^2 into a connected sum of Moebius bands with the only T -band and several quaternion bands. By the transfer homomorphism M^2 is covered by (a non-oriented) surface \tilde{M}^2 . A T -band in the decomposition of M^2 is transformed into a cylinder on \tilde{M}^2 , the generator

$\tilde{l} \subset \tilde{M}^2$ of the cylinder is a closed loop on corresponds to the double covering over the generator $l \subset M^2$ of the T -band, the Hopf invariant $h(\tilde{l}) \in \mathbb{Z}/2$ of \tilde{l} loop is trivial. The each quaternion band on M^2 is covered by a pair of quaternion mirror-symmetric bands on \tilde{M}^2 . This proves that the image $(M^2, \eta_M, \Xi_M)^!$ in V is trivial. Lemma 5 is proved.

The last part of the proof of Lemma 4 is following. Let (M^2, η_M, Ξ_M) represents an arbitrary element in $W_{\mathbf{Q}}$. Consider the manifold P^3 with boundary $\partial P^3 = M^2$, the manifold is equipped with a normal bundle structure (P^3, ζ_P, Ψ_P) , this structure determines a boundary of the form (M^2, η_M, Ξ_M) . Denote by $Q^2 \subset P^3$ a surface, which is defined as a dual surface to ζ_β . Obviously, there exists a closed characteristic surface, because $\zeta_{\beta;P}|_{\partial P^3}$ is null-homotopic. Then $(Q^2, \zeta_P|_Q, \Psi_P|_Q)$ determines an element $x \in W$, the transfer $x^!$ belongs to $\text{Ker}(\rho^\otimes)$ by Lemma 5. By the construction, $\rho^\otimes[(M^2, \eta_M, \Xi_M)]$ coincides with $x^! = (Q^2, \zeta_P|_Q)^!$ in V . Lemma 4 is proved.

Proof of Theorem 5

Let us define a hyperquaternionic form (M^2, η_M, Ξ_M) . Consider a pair of Moebius bands $(\mu_i, \partial) \subset M^2, i = 1, 2$, the generators of μ_1, μ_2 is represented by η_M into the elements TJ, T correspondingly. The connected sum $(\mu_1, \partial) \sharp (\mu_2, \partial)$ along the common boundary $\partial\mu_1 = \partial\mu_2$ coincides to M^2 . The characteristic mapping η_M admits a reduction: $\eta_M : M^2 \rightarrow K(\mathbf{D}, 1) \subset K(\mathbb{N}, 1)$.

By the construction, M^2 contains a thin cylinder $C_J \subset M^2$, the (orientation preserved) loop $l_J \subset C_J$ which corresponds to the element $J \in \mathbf{D} \subset \mathbb{N}$ by η_M . The surface $M^2 \setminus C_J$ is diffeomorphic to the cylinder C_{-J} , this cylinder is a non-oriented cycle between the two copies of ∂C_J . Denote the segment of the cylinder C_{-J} , which is transversal to the central line of C_{-J} by $l_T \subset C_{-T}$. Extend the segment $l_T \subset C_{-T} \subset M^2$ by a closed loop on M^2 by a short path, which is transversal to l_J . This closed path is denoted by $l_T \subset M^2$. The closed path $l_{JT} \subset M^2$ are defined as the central path in $M^2 \setminus l_T$. The paths l_T, l_{JT} coincide with central lines the the Moebius bands μ_1, μ_2 on M^2 .

The (orientation reversed) loop $l_T \subset M^2$ corresponds to the element $T \in \mathbf{D} \subset \mathbb{N}$ by η_M . The element $\eta_M(l_T^{-1} \circ l_J \circ l_T \circ l_J)$ is the trivial element in $\mathbf{D} \subset \mathbb{N}$, because $[T, J] = -1$. Informally speaking, the Klein bottle M^2 is the result of a non-oriented self-homology of l_J by l_T .

Describe a regular cobordism of $2(M^2, \eta_M, \Xi_M)$ into a form (L^2, η_L, Ξ_L) , where L^2 is the Klein bottle, which is defined analogously to M^2 . Take the orientation preserving loop $l_{-1} \subset C_{-1} \subset L^2$, which represents the element $J^2 = -1 \in \mathbf{D} \subset \aleph$ by η_L . The loop l_{-1} is the analog of the loop $l_J \subset M^2$. Define the orientation reversed loop, which is analog of the loop $l_T \subset M^2$. Denote the corresponding cycle of l_{-1} by S_1 , denote the corresponding cycle of l_T by S_2 .

Lemma 6. *The form $2(M^2, \eta_M, \Xi_M)$ is equivalent to the form (L^2, η_L, Ξ_L) (probably, up to an element of the order 2 in W).*

Describe a regular cobordism of $2(L^2, \eta_L, \Xi_L)$ into a form (K^2, η_K, Ξ_K) , where K^2 is the Klein bottle, as in the case of M^2 and L^2 . Denote the orientation preserved cycle $R_2 \subset C \subset K^2$, which is the analog of the cycle $S_2 \subset C_{-1} \subset L^2$ and which is represented into the trivial element in \aleph , by η_K . In this formula C in a thin cylinder, which is the analog of the cylinder C_{-1} . Denote the orientation reversed cycle $l_T \subset \mu_1 \subset K^2$ by R_1 .

Recall that an immersion $f : K^2 \looparrowright \mathbb{R}^{10}$ with the prescribed isomorphism $\Xi_K : \nu_K \cong \eta_K^*(A) \oplus \eta_{\alpha;K}^*(\gamma) \oplus 3\eta_{\beta;K}^*(\gamma)$ of the normal bundle, where A is the universal 4-bundle over the subspace $K(\mathbb{Z}/2(-1) \oplus \mathbb{Z}/2(T), 1) \subset K(\aleph, 1)$, γ is the universal line bundle, is well-defined. The element $\eta_K(R_1)$ is the trivial element in \aleph . Moreover, the mapping $\eta_K(R_1)$ has the target a point in $K(\aleph, 1)$.

The curve $f(R_1)$ is a framed curve in \mathbb{R}^{10} and the stable Hopf invariant $h(R_1) \in \mathbb{Z}/2 = \{0, 1\}$ is well-defined.

Lemma 7. *The form $2(L^2, \eta_L, \Xi_L)$ is equivalent, probably, up to an element of the order 2 in W , to a form (K^2, η_K, Ξ_K) , where the oriented framed loop R_1 has the Hopf invariant $h(R_1) \neq 0, h(R_1) \in \mathbb{Z}/2$.*

Proof of Lemma 6 and Lemma 7

Proofs of Lemmas are analogous. Let us prove Lemma 7. The characteristic mapping η_L takes the image in the subgroup $\mathbb{Z}/2(-1) \times \mathbb{Z}/2(T) \subset \aleph$, where the generators of the factors are $\{-1, T\}$.

Define the normal bundle structure Ξ_L as following. The normal bundle for (L^2, η_L, Ξ_L) is represented by a Whitney sum of 4-bundle, 3-bundle and the trivial line bundle $A \oplus B \oplus \varepsilon$.

The bundle B is splitted into the Whitney sum of 3 isomorphic line bundles: $B = B_1 \oplus B_2 \oplus B_3$. Each factor B_j , $j = 1, 2, 3$ is the line bundle, which is skew along the cycle R_2 by means of the element T , and is constant along the cycle R_1 . The factors correspond to $\eta_{\beta;L}^*(\gamma)$.

The bundle A is splitted into the Whitney sum of 2 isomorphic copies of plane-bundles: $A = A_1 \oplus A_2$. The plane bundle A_1 (and A_2) should be looked as a line complex bundle. The each line complex bundle is equipped with the Hermitian conjugation long the cycle R_2 by means of the point symmetry, given by multiplication on -1 along the cycle R_1 . The factors A_1, A_2 are inside the 4-dimensional block $\eta_L^*(A)$ of ν_{L_2} with generators $\{\pm 1, T\}$.

The factor ε corresponds to $\eta_{\alpha;L}^*(\gamma)$.

Denote two copies of (L^2, η_L, Ξ_L) by $(L_1^2, \eta_{L_1}, \Xi_{L_1})$, $(L_2^2, \eta_{L_2}, \Xi_{L_2})$. Define the following form $(L_2^2, \eta_{L_2}^{op}, \Xi_{L_2}^{op})$, which represents an element in W . The characteristic classes $\eta_{L_2}, \eta_{L_2}^{op}$ coincide, the normal bundle structure $\Xi_{L_2}^{op}$ is derived from Ξ_{L_2} by the reversing of the orientation of the each factors $B = B_1 \oplus B_2 \oplus B_3$ and by the complex conjugation on the factors A_1, A_2 . In particular, the local orientations on the surfaces (L_2^2, Ξ_{L_2}) and $(L_2^2, \Xi_{L_2}^{op})$ with a prescribed normal bundle structure are opposite.

Let us prove that the forms $(L_2^2, \eta_{L_2}, \Xi_{L_2})$, $(L_2^2, \eta_{L_2}, \Xi_{L_2}^{op})$ are equivalent in W . Take a self-homotopy of η_{L_2} into itself such that the trace of a point $pt \in L_2$ by this homotopy represents the generator $T \in \mathbb{Z}/2(-1) \times \mathbb{Z}/2(T) \subset \aleph$. By this homotopy the framing Ξ_{L_2} is transformed into a framing $\Xi_{L_2}^{op}$, where $\Xi_{L_2}^{op}$ is the composition of Ξ_{L_2} with the reflection in the factors B_1, B_2, B_3, A_1, A_2 as described above. The forms are equivalent.

Let us prove that the form $(L_1^2, \eta_{L_1}, \Xi_{L_1}) \cup (L_2^2, \eta_{L_2}, \Xi_{L_2}^{op})$ is regular cobordant to the form (K^2, η_K, Ξ_K) , probably, up to a form (P^2, η_P, Ξ_P) with the characteristic class η_P takes the image in the central subgroup $\mathbb{Z}/2(-1) \subset \aleph$.

Take the restriction of Ξ_1 and of Ξ_2^{op} over the cycle $R_1 \subset L_1^2$ and the cycle $-R_1' \subset L_2^2$ correspondingly (in this formula $-R_1'$ is the cycle on L_2^2 which corresponds to the cycle R_1 with the opposite orientation, using the diffeomorphism $L_1^2 \cong L_2^2$). The restrictions $\Xi_1|_{R_1}, \Xi_2^{tw}|_{-R_1}$ are 4-dimensional $(-1, T)$ -framings, which are stabilized in the codimension 4 by corresponding framings on $B \oplus \varepsilon$ ($B_i|_{R_1}, i = 1, 2, 3$ is the trivial bundle, the trivialization Ξ_2 over $B|_{R_1}$ is opposite to the trivialization $\Xi_2^{tw}|_{R_1}$, the trivialization Ξ_2 over $A|_{R_1}$ is conjugate to the trivialization $\Xi_2^{tw}|_{R_1}$).

Denote a $(-1, T)$ -framing Ξ_2^{tw} over (L_2, η_{L_2}) as following. Denote the line subbundles $\lambda_1 \subset A_1, \lambda_2 \subset A_2$, which correspond to the imaginary axis of the complex line bundles. The line bundle λ over L_2^2 is well-defined, and this bundle is skew over the cycle $R_2 \subset L_2^2$, which corresponds to the element T . Take the rotation trough the angle π inside the 4-bundle $B \oplus \lambda$ over L_2 . As the result we get the new $(-1, T)$ -framing over L_2^2 , denoted by Ξ_2^{tw} . The framing Ξ_2^{tw} coincides to the framing Ξ_1 everywhere, except the line bundle $\lambda_2 \subset \nu_{L_1}$, on this factor the framing Ξ_2^{tw} is given by the reflection of Ξ_1 . The framings Ξ_2^{tw} is equivalent to the framing Ξ_2^{op} , and is equivalent to the framing Ξ_1 .

Assume without loss of a generality that the restriction of the framing $\Xi_1|_{R_1}$ to the subbundle $B \oplus \varepsilon$ over the cycle R_1 is parallel to the coordinate axis e_7, e_8, e_9, e_{10} . Assume the framing $\Xi_1|_{R_1}$ on the factors λ_1, λ_2 is parallel to the vectors e_4, e_6 correspondingly. Assume the framing $\Xi_1|_{R_1}$ on the factors A_1, A_2 is parallel to the vectors $(e_3, e_4), (e_5, e_6)$ correspondingly. Then the skew-framing $\Xi_2^{tw}|_{R_1}$ coincides to the Ξ_1 along each directions, but the direction of the coordinate vector e_6 , where Ξ_1, Ξ_2^{tw} are opposite.

The -1 -structure of skew framings $\Xi_1|_{R_1}, \Xi_2^{tw}|_{-R'_1}$ are distinguished only inside the factor A_2 of the normal bundle of $L_1^2 \cong L_2^2$, by a full rotation trough the angle 2π .

Take the regular cobordism transformation of the form $(L_1^2, \eta_{L_1}, \Xi_{L_1}) \cup (L_2^2, \eta_{L_2}, \Xi_{L_2}^{tw})$ by a surgery, with a support in small neighborhoods of a corresponding pair of points on $R_1, -R'_1$. As the result we get the form $(L_4^2, \eta_{L_4}, \Xi_{L_4})$. The image of η_{L_4} is in the space $K(\mathbb{Z}/2(-1) \times \mathbb{Z}/2(T), 1)$. The cycle $R_1 \cup -R'_1$ is transformed into a cycle $R_3 \subset L_3^2$. The image of the characteristic class $\eta_{L_3}(R_3)$ is null-homotopic in the target space $K(\mathbb{Z}/2(-1) \times \mathbb{Z}/2(T), 1)$. The stable Hopf invariant $h(R_3)$ of the framed curve R_3 is non-trivial.

The form $(L_3^2, \eta_{L_3}, \Xi_{L_3}) \cup (K^2, \eta_K, \Xi_K)$ is cobordant to a form $(L_4^2, \eta_{L_4}, \Xi_{L_4})$, where the image of the mapping η_{L_4} is inside the space $K(\mathbb{Z}/2(-1), 1)$. The form $(L_4^2, \eta_{L_4}, \Xi_{L_4})$ is trivial, or, is of the order 2 in W . Theorem 5 is proved.

Conclusion

V.I.Arnol'd formulated the problem [[Arn], Problem 1984-12]: "To transform asymptotic ergodic definition of the Hopf invariant of divergence-free vector fields to the theory of S.P.Novikov, which generalize the Whitehead product of homotopy groups of spheres".

Algebraic commutators, which are used to define the higher invariants of classical links, are particular Whitehead products in homotopy groups of spheres. M -invariant is a special generalized Whitehead product, which admits asymptotic and ergodic property. To keep additional symmetry of magnetic fields we have to apply the M -invariant for links in various homogeneous manifolds, which are rational homology spheres. For classical links M -invariant is associated with the Arf-invariant in the stable homotopy group Π_2 . Hypothetic modifications of M -invariant for links in S^3/\mathbb{Q} are associated with Arf-Brown invariant in the stable homotopy group Π_3 , and with hyperquaternionic Arf-invariant in the stable homotopy group Π_7 . The constructions give a solution (in part) of the Arnol'd Problem.

References

- [Arn] Arnol'd V.I. *Arnol'd problems*, Moscow, Phasis (2000).
- [A-Kh] Arnol'd V.I., Khesin B.A., *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer (1998). (in Russian: Moscow, MCNMO (2007)).
- [A] Akhmet'ev, P.M. *Quadratic magnetic helicity and magnetic energy*, Proc. Steklov Math. Inst. 278 (2012) 16-28.
- [A2] Akhmet'ev, P.M. *On a higher integral invariant for closed magnetic lines*, Journal of Geometry and Physics **74** (2013) 381-391.
- [A3] Akhmet'ev, P.M. *On combinatorial properties of a higher asymptotic ergodic invariant of magnetic lines*, Journal of Physics: Conference Series 544 (2014) 012015.
- [A4] Akhmet'ev P.M. *On an problem by V.I.Arnol'd about higher analog of asymptotic Hopf invariant*, Problemi Matematicheskogo Analiza **79** (2015) 33-42.
- [A-E] P.M.Akhmet'ev, P.J.Eccles, *The relationship between framed bordism and skew-framed bordism*, Bull. Lond. Math. Soc. 39, No. 3 (2007) 473-481.
- [A-K-K] P. M. Akhmet'ev, O. V. Kunakovskaya, V. A. Kutvitskii, *Remark on the dissipation of the magnetic helicity integral*, Teoret. Mat. Fiz., 158:1 (2009) 150–160.
- [B-F] Berger, M.A. and Field, G.B. *The topological properties of magnetic helicity*, J. Fluid Mech. 147 (1984) 133-148.
- [C-M] H.S.M. Coxeter, W.O.J. Moser *Generators and Relations for Discrete Groups*, Springer-Verlag (1972).
- [G-M] Progress in Mathematics, Vol. 62, *A la recherche de la Topologie Perdue*, Birkhäuser (1986).
- [K] F.Klein, *Vorlesungen Über das Ikosaeder und die Auflösung der Gleichungen von Funften Grade*, Leipzig (1884).
- [M-P] Matveev, S. & Polyak, M. *A simple formula for the Casson-Walker invariant*, Journal of Knot Theory and Its Ramifications **18** 6 (2009) 841–864; arXiv:0811.0606.

- [M] Melikhov, S.A. *Colored finite type invariants and a multi-variable analogue of the Conway polynomial*, Preprint (2003) arXiv:math/0312007v2.
- [M] Moffatt, H.K. *The topology of turbulence*, Lesieur, M. (ed.) et al., Ecole de Physique des Houches – UJF INPG – Grenoble, a NATO Advanced Study Institute. Les Houches session LXXIV 31 July – 1 September 2000, New trends in turbulence. Turbulence: nouveaux aspects. Berlin: Springer; Les Ulis: EDP Sciences (2001) 321-340.
- [M-R] H.K.Moffatt & R.L. Ricca, *Helicity and Călugăreanu invariant*, Proc. R. Soc. Lond. A. **439** (1992) 411-429.
- [R] Rädler K.-H. *The generation of cosmic magnetic fields// From the Sun to the Great Attractor/*, Guanajuato Lect. on Astrophysics (D. Page and J. G. Hirsh, eds.) Lect. Notes Phys.—Springer-Verlag (1999) 101-172.
- [S-S] V. B. Semikoz & D. Sokoloff, *Magnetic Helicity and Cosmological Magnetic Field*, Astron. Astrophys. 433 (3) (2005) L53–L56.
- [S] P. Scott, *The Geometry of 3-Manifolds*, The Bulletin of the London Mathematical Society Vol 15, Part 5, (1983), N56, pp. 401-487.
- [Y-H] A. R. Yeates & G. Hornig *Unique Topological Characterization of Braided Magnetic Fields*, Journal of Physics: Conference Series 544 (2014) 012002, arXiv:1208.2286v1 (2012) 10 Aug.